finite element method for 1d continua

- motivation - spatial truss structures
  - real structures
  - mechanical modeling
  - weak form
- illustration of fem by means of an example
- computer oriented development of finite element method
- element tensors and element matrices
- structural equation of motion or structural static equilibrium
- solution and post-processing
- examples
• model of a tensegrity exposition structure
  KUHL ET AL. (2011)
  o compression truss elements
  o model of the structure
  o detail of the structure
• mechanical model
  o 1d continuum
  o spatial truss structure

information about title picture ’1d fem’
motivation - towers of wind power plants
loading of wind power plant towers

- loading of tower from gondola
  - dead load of gondola, rotor and hub (negative) $F_1$
  - transverse force caused by cross wind $F_2$
  - transverse force caused by rotor drag $F_3$
  - yaw moment caused by non-symmetric wind $M_1$
  - pitch moment caused by dead load of rotor and hub $M_2$
  - rolling moment caused by rotor and generator moment $M_3$

- direct loading of tower
  - dead load (negative) $b_1, p_1$
  - transverse force caused by wind $q_2, q_3$

- individual study of loading with normal forces
  - dead load (negative) $b_1, p_1$
  - single load from gondola $F_1$

motivation - towers of wind power plants
motivation - towers of wind power plants
normal loading

dead load $p_1$

cross section $A$

construction of tower

cross section $A(X_1)$

F_1

nor mal loading

motivation - towers of wind power plants
dead load \( p_1 \) caused by dead load

- constant cross section area \( A = \text{constant} \)
- line force \( F_1 = -\rho g A \)
- volume force of one dimensional continua \( b_1 = p_1 \rho A = -g \)

- single load at tower head (negative)

- NEUMANN boundary condition - traction \( t^\star_1 = F_1 A \)

- model problem with constant line force
  - volume force and load parameter \( b_1 = b_1 \) with \( b = -g \)

linear computational structural mechanics

2 1d finite element method

2.1 motivation - 1d structures

motivation - towers of wind power plants
normal loading
tower with a (almost) constant cross section
e.g. enercon e 36

- line load caused by dead load
  - constant cross section area
    \[ A = \text{constant} \]
  - line force
    \[ p_1 = -\rho g A \]
  - volume force of one dimensional continua
    \[ b_1 = \frac{p_1}{\rho A} = -g \]

- single load at tower head (negative)
  - \text{NEUMANN} boundary condition - traction
    \[ t^*_1 = \frac{F_1}{A} \]
tower with a (almost) constant cross section
e.g. enercon e 36

- line load caused by dead load
  - constant cross section area
    \[ A = \text{constant} \]
  - line force
    \[ p_1 = -\rho g A \]
  - volume force of one dimensional continua
    \[ b_1 = \frac{p_1}{\rho A} = -g \]

- single load at tower head (negative)
  - \text{NEUMANN} boundary condition - traction
    \[ t_1^* = \frac{F_1}{A} \]

- model problem with constant line force
  - volume force and load parameter
    \[ b_1 = b \quad \text{with} \quad b = -g \]
• rotor blade coordinate system
  ◦ $e_1$ - in direction of rotor blade
  ◦ $e_2$ - in direction of generator shaft
  ◦ $e_3 = e_1 \times e_2$ - tangential direction

• centrifugal force of volume element $dV$
  ◦ mass of volume element $dm = \rho dV$

• number of revolutions of rotor $n$
  ◦ angular velocity of rotor $\dot{\phi}_2 = 2\pi n$

• distance generator shaft center and volume element $X_1$

• centrifugal force of volume element $dF_1 = \dot{\phi}_2^2 X_1 \rho dX_2 dX_3$

• resulting line force $p_1(A)$

\[
p_1(X_1) = \int_A \dot{\phi}_2^2 X_1 \rho dX_2 dX_3 = \dot{\phi}_2^2 X_1 \rho A(X_1)
\]

• comment: also other relevant forces (dead load, wind loads) are acting on the rotor blade (not considered here)
rotor blade coordinate system

- $e_1$ - in direction of rotor blade
- $e_2$ - in direction of generator shaft
- $e_3 = e_1 \times e_2$ - tangential direction

centrifugal force of volume element $dF$

mass of volume element $dm = \rho dV$

number of revolutions of rotor $n$

angular velocity of rotor $\dot{\phi}_2 = \frac{2\pi n}{2}$

distance generator shaft center and volume element $X_1$

centrifugal force of volume element $dF$

resulting line force $p_1$

\[
p_1(X_1) = \int_A \dot{\phi}_2 \rho dX_2 dX_3 = \dot{\phi}_2 \rho A(X_1)
\]

comment: also other relevant forces (dead load, wind loads) are acting on the rotor blade (not considered here)

motivation - rotor blades of wind power plants
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• rotor blade coordinate system
  ◦ $e_1$ - in direction of rotor blade
  ◦ $e_2$ - in direction of generator shaft
  ◦ $e_3 = e_1 \times e_2$ - tangential direction

• centrifugal force of volume element $dF$
 ◦ mass of volume element $dm = \rho dV$
 ◦ number of revolutions of rotor $n$
 ◦ angular velocity of rotor $\dot{\varphi}$
  $2 \pi n$
  ◦ distance generator shaft center and volume element $X$

• centrifugal force of volume element $dF_1$
  $\dot{\varphi} X dF_1 = \rho dV$

• resulting line force $p_1$
  $p_1(X_1) = \int_{A} \dot{\varphi} \cdot X_1 \rho dX_2 dX_3$

• comment: also other relevant forces (dead load, wind loads) are acting on the rotor blade (not considered here)

motivation - rotor blades of wind power plants
• rotor blade coordinate system
  ○ $e_1$ - in direction of rotor blade
  ○ $e_2$ - in direction of generator shaft
  ○ $e_3 = e_1 \times e_2$ - tangential direction
• centrifugal force of volume element $dV$
  ○ mass of volume element
    \[ dm = \rho dV \]
  ○ number of revolutions of rotor $n$
  ○ angular velocity of rotor $\dot{\phi}_2 = 2\pi n$
  ○ distance generator shaft center and volume element $X_1$
  ○ centrifugal force of volume element
    \[ dF_1 = \dot{\phi}_2^2 X_1 \rho dV \]
• resulting line force $p_1$ (integration over $A$)
  \[ p_1(X_1) = \int_A \dot{\phi}_2^2 X_1 \rho dX_2 dX_3 = \dot{\phi}_2^2 X_1 \rho A(X_1) \]
• comment: also other relevant forces (dead load, wind loads) are acting on the rotor blade (not considered here)
- vector formulation of centrifugal force
  - distance vector generator shaft - volume element $x$
  - angular velocity vector of rotor
    $\dot{\phi} = [\dot{\phi}_1 \ \dot{\phi}_2 \ \dot{\phi}_3]^T$
  - tangential velocity volume element
    $\dot{u} = \dot{\phi} \times X$
  - acceleration volume element (constant rotation axis $\dot{\phi}/||\dot{\phi}||$ and angular velocity $||\dot{\phi}||$)
    $\ddot{u} = \frac{\partial \dot{u}}{\partial t} = -\dot{\phi} \times [\dot{\phi} \times X]$
  - centrifugal force volume element
    $dF = -\dot{\phi} \times [\dot{\phi} \times X] \rho dV$

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**motivation - rotor blades of wind power plants**
simplified rotor blade with a constant cross section

e.g. SMITH-PUTNAM wind power plant

- line load caused by centrifugal force
  - constant cross section area
    \[ A = \text{constant} \]
  - line force
    \[ p_1(X_1) = \dot{\varphi}_2^2 \rho A X_1 \]
  - volume force of one dimensional continua
    \[ b_1(X_1) = \frac{p_1(X_1)}{\rho A} = \dot{\varphi}_2^2 X_1 \]

PUTMAN (1948): 'Power from the Wind'
simplified rotor blade with a constant cross section
e.g. SMITH-PUTNAM wind power plant

- line load caused by centrifugal force
  - constant cross section area
    \[ A = \text{constant} \]
  - line force
    \[ p_1(X_1) = \varphi_2^2 \rho A X_1 \]
  - volume force of one dimensional continua
    \[ b_1(X_1) = \frac{p_1(X_1)}{\rho A} = \varphi_2^2 X_1 \]

- model problem with linearly changing line force
  - volume force and load parameter
    \[ b_1(X_1) = \frac{2b}{L} X_1 \quad \text{with} \quad b = \frac{\varphi_2^2 L}{2} \]

PUTMAN (1948): 'Power from the Wind'
columns, railway station Willhelmshöhe, Kassel

rod model for structural analysis
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rod model for structural analysis
smokestack, university of kassel
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rod model for structural analysis
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normal loading

\[ F_1 = mg \]

dead load \( p_1(X_1) \)

cross section \( A(X_1) \)

column

rod model for structural analysis
normal loading

transverse loading

- transverse loading and transverse deformation

- and normal force and normal deformation

- can independently calculated (see beam models)
normal loading

\[ F_1 = mg \]

\[ p_1(X_1) \]

cross section \( A(X_1) \)

dead load \( p_1(X_1) \)

\[ e_3 \]

\[ e_1 \]

\[ e_3 \]

column

\[ F_1 \]

\[ p_1 = \text{constant} \]

\[ A = \text{constant} \]

rod model for structural analysis
normal loading

\[ F_1 = mg \]

dead load \( p_1(X_1) \)

cross section \( A(X_1) \)

dead load \( p_1(X_1) \)

cross section \( A(X_1) \)

column

\[ F_1 \]

\( p_1 = \text{constant} \)

\[ A = \text{constant} \]

truss element

\[ F_1 \]

\( p_1 \approx \text{small} \)

\[ A = \text{constant} \]

rod model for structural analysis
normal loading

\[ F_1 = mg \]

cross section \( A(X_1) \)

defad load \( p_1(X_1) \)

column

\[ p_1 = \text{constant} \]

\[ A = \text{constant} \]

truss element

\[ p_1 \approx 0 \]

\[ A = \text{constant} \]
• application of rod model for structural analysis of
  
  ○ columns, stanchions

  ○ masts, towers, smokestacks

  ○ pylons

  ○ towers and rotor blades of wind turbines

• transverse loading and transverse deformation / normal force / normal deformation can independently calculated (see beam models)

• truss element is included as special case

  ○ dead load of the bar is negligible compared with nodal forces

  ○ analysis of spatial and planar truss structures (forces and deformation)
motivation - lattice tower of wind power plants
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motivation - lattice tower of wind power plants
balance of linear momentum, CAUCHY equation of motion for $X \in \Omega$ ($N_1 = \sigma_{11} A$, $p_1 = \rho b_1 A$)

$\rho \ddot{u}_1 = \sigma_{11,1} + \rho b_1$ \hspace{1cm} $\rho A \ddot{u}_1 = N_{1,1} + p_1$

**DIRICHLET boundary conditions for $X \in \Gamma_u$**

$u_1 = u_1^*$

**NEUMANN boundary conditions for $X \in \Gamma_\sigma$**

($N_1 = \sigma_{11} A$, $n_1 = 1$)

$\sigma_{11} n_1 = t_1^*$ \hspace{1cm} $N_1 = N_{1}^*$

**initial conditions for $X \in \Omega$**

$u_1(t_0) = u_{10}$, $\dot{u}_1(t_0) = \ddot{u}_{10}$

**1d continuum & equilibrium**

balance mom. $\rho \ddot{u}_1 = \sigma_{11,1} + \rho b_1$

constitutive law $\sigma_{11} = E \varepsilon_{11}$

kinematics $\varepsilon_{11} = u_{1,1}$

**mechanics of one-dimensional continua**
- differential equation within domain $\Omega$

\[
\rho \ddot{u}_1 - \rho b_1 = \sigma_{11,1} \quad \text{kinetics}
\]

\[
\sigma_{11} = E \varepsilon_{11} \quad \text{constitutive law}
\]

\[
\varepsilon_{11} = u_{1,1} \quad \text{kinematics}
\]

\[
\rho \ddot{u}_1 - \rho b_1 = E u_{1,11} \quad \forall \ X_1 \in \Omega
\]

- boundary conditions at DIRICHLET $\Gamma_u$ and NEUMANN $\Gamma_\sigma$ boundaries

\[
u_1 = u^*_1 \quad \forall \ X_1 \in \Gamma_u
\]

\[
\sigma_{11} n_1 = \sigma_{11} = t^*_1 \quad \forall \ X_1 \in \Gamma_\sigma
\]

- initial conditions for displacement - or acceleration field

\[
u_1(t_0) = u_0 \quad \forall \ X_1 \in \Omega
\]

\[
\dot{u}_1(t_0) = \ddot{u}_1 0 \quad \forall \ X_1 \in \Omega
\]

- characterization
  - linear partial differential equation
  - with respect to space $(u_{1,11})$ and time $(\ddot{u}_1)$

**strong form - initial boundary value problem**
The initial boundary-value problem of elastodynamics is illustrated in the diagram. The kinetics part includes stresses $\sigma_{11}(X_1, t), \sigma_{11}^0(X_1)$ and the balance of linear momentum $\rho \ddot{u}_1 = \sigma_{11,1} + \rho b_1$. The material part features the constitutive law $\sigma_{11} = E \varepsilon_{11} + \sigma_{11}^0$, and the strains $\varepsilon_{11}(X_1, t), \varepsilon_{11}^\theta(X_1, t)$. The kinematics section shows the strains $\varepsilon_{11} = u_{1,1} - \varepsilon_{11}^\theta$, deformation $\varepsilon_{11} = u_{1,1} - \varepsilon_{11}^\theta$, and initial conditions $u_1(X_1, t_0), (\ddot{u}_1(X_1, t_0))$.

Boundary conditions are also specified: Neumann $\sigma_1 n_1 = t_1^*$ and Dirichlet $u_1 = u_1^*$. Initial stresses $\sigma_{11}^0(X_1) = \sigma_{11}(X_1, 0)$ are initial conditions for the system. The body $\Omega$ is subject to load $\rho b_1$ and thermal strains $\varepsilon_{11}^\theta = \alpha [\theta - \theta_{\text{ref}}]$. The TONTI scheme and Stein&Barthold (1996) are mentioned for further reading.
line load

\[ p_1(X_1) = -N_{1,1}(X_1) \]

equilibrium, balance of linear momentum

normal force

\[ N_1(X_1) = A\sigma_{11} = EA\varepsilon_{11} = EAu_{1,1} \]

constitutive law and kinematics

normal displacement

\[ u_1(X_1) = C_1 + \int p_1(X_1) \, dX_1 \]

displacement field

integration of deformation of bar (statics)
line load
\[ N_1(X_1) = - \int p_1(X_1) \, dX_1 + C_1^1 \]

normal force
\[ p_1(X_1) = -N_{1,1}(X_1) \]

normal displacement
\[ N_1(X_1) = A\sigma_{11} = EA\epsilon_{11} = EAu_{1,1} \]

constitutive law and kinematics

equilibrium, balance of linear momentum

displacement field

integration of deformation of bar (statics)
line load

\[ N_1(X_1) = -\int p_1(X_1) \, dX_1 + C_1^1 \]

normal force

\[ u_1(X_1) = \frac{1}{EA} \int N_1(X_1) \, dX_1 + C_1^2 \]

equilibrium, balance of linear momentum

\[ p_1(X_1) = -N_{1,1}(X_1) \]

constitutive law and kinematics

\[ N_1(X_1) = A\sigma_{11} = EA\varepsilon_{11} = E A u_{1,1} \]

displacement field

+ Neumann and Dirichlet boundary conditions for determination of \( C_1^1 \) and \( C_1^2 \)
motivation - tower

model problem - tension bar loaded by $\rho b_1$ (without $F_1$)

$\rho b_1, b_1 = b = \text{constant}$

$E, A, \rho, \Omega = [0, L]$  

$L$

- balance of momentum
  (plus constitutive law and kinematics)

$$E u_{1,11} + \rho b_1 = 0$$

- homogeneous DIRICHLET boundary conditions

$$u_1(X_1 = 0) = 0$$

- NEUMANN boundary condition (traction free surface)

$$t_1^* = \sigma_{11}(X_1 = L) \ n_1 = \sigma_{11}(X_1 = L) = 0$$

$$\sigma_{11}(X_1 = L) = E \ u_{1,1}(X_1 = L) \rightarrow u_{1,1}(X_1 = L) = 0$$

example - tension bar
solution of boundary value problem

- integration of balance of linear momentum
  \[
  \frac{\partial u_1(X_1)}{\partial X_1} = -\frac{\rho b}{E} \int dX_1 + C_1^1 = -\frac{\rho b}{E} X_1 + C_1^1
  \]

- NEUMANN boundary condition
  \[
  \frac{\partial u_1(X_1 = L)}{\partial X_1} = -\frac{\rho b}{E} L + C_1^1 = 0 \quad \rightarrow \quad C_1^1 = \frac{\rho b}{E} L
  \]

- normal strain - spatial derivative of displacement
  \[
  \frac{\partial u_1(X_1)}{\partial X_1} = \varepsilon_{11}(X_1) = \frac{\rho b}{E} [L - X_1]
  \]

- integration of normal strain (spiration and integration)
  \[
  u_1(X_1) = \frac{\rho b}{E} \left[ \int L \, dX_1 - \int X_1 \, dX_1 \right] + C_2^1
  = \frac{\rho b}{E} \left[ L \, X_1 - \frac{1}{2} \, X_1^2 \right] + C_2^1
  \]

- DIRICHLET boundary condition
  \[
  u_1(X_1 = 0) = C_1^2 = 0 \quad \rightarrow \quad C_1^2 = 0
  \]
tension bar loaded by $\rho b_1$

$\rho b_1, b_1 = b = \text{constant}$

solution of boundary value problem

- displacement field

$$u_1(X_1) = \frac{\rho b}{E} \left[ L X_1 - \frac{1}{2} X_1^2 \right]$$

- displacements at selected positions

$$u_1(L) = \frac{\rho b}{E} \left[ L^2 - \frac{1}{2} L^2 \right] = \frac{1}{2} \frac{\rho b L^2}{E}$$

$$u_1\left(\frac{3L}{4}\right) = \frac{\rho b}{E} \left[ \frac{3}{4} L^2 - \frac{9}{32} L^2 \right] = \frac{15}{32} \frac{\rho b L^2}{E}$$

$$u_1\left(\frac{2L}{3}\right) = \frac{\rho b}{E} \left[ \frac{2}{3} L^2 - \frac{4}{18} L^2 \right] = \frac{4}{9} \frac{\rho b L^2}{E}$$

$$u_1\left(\frac{L}{2}\right) = \frac{\rho b}{E} \left[ \frac{1}{2} L^2 - \frac{1}{8} L^2 \right] = \frac{3}{8} \frac{\rho b L^2}{E}$$

$$u_1\left(\frac{L}{3}\right) = \frac{\rho b}{E} \left[ \frac{1}{3} L^2 - \frac{1}{18} L^2 \right] = \frac{5}{18} \frac{\rho b L^2}{E}$$

$$u_1\left(\frac{L}{4}\right) = \frac{\rho b}{E} \left[ \frac{1}{4} L^2 - \frac{1}{32} L^2 \right] = \frac{7}{32} \frac{\rho b L^2}{E}$$

d.kuhl, wes.online, university of kassel linear computational structural mechanics 2 1d finite element method 2.2 modeling - strong form

example - tension bar (load case i)
tension bar loaded by $\rho b_1$

- displacement field
  \[ u_1(X_1) = \frac{\rho b}{E} \left[ L X_1 - \frac{1}{2} X_1^2 \right] \]
- strain field - derivative of displacement field
  \[ \varepsilon_{11}(X_1) = u_{1,1}(X_1) = \frac{\rho b}{E} [L - X_1] \]
- stress field - constitutive law
  \[ \sigma_{11}(X_1) = E \varepsilon_{11} = \rho b [L - X_1] \]
- derivative of stress field
  \[ \sigma_{11,1}(X_1) = E \varepsilon_{11,1} = -\rho b \]
- residuum (local error)
  solution of differential equation
  \[ \sigma_{11,1}(X_1) + \rho b_1 = -\rho b + \rho b = 0 \]
  local error is zero $\rightarrow$ analytical solution is correct!

example - tension bar (load case i)
\[ \frac{u_1 E}{\rho b L^2} \]

\[ \frac{\varepsilon_{11} E}{\rho b L} \]

\[ \frac{\sigma_{11}}{\rho b L} \]

\[ \frac{\sigma_{11,1} + \rho b_1}{\rho b} \]
balance of momentum and NEUMANN boundary condition

\[ \rho \ddot{u}_1 - \rho b_1 - \sigma_{11,1} = 0 \quad \forall X_1 \in \Omega \]
\[ \sigma_{11} n_1 - t^*_1 = 0 \quad \forall X_1 \in \Gamma_\sigma \]

multiplication with virtual displacement \( \delta u_1 \)

\[ \delta u_1 \left[ \rho \ddot{u}_1 - \rho b_1 - \sigma_{11,1} \right] = 0 \]
\[ \delta u_1 \left[ \sigma_{11} n_1 - t^*_1 \right] = 0 \]

properties of virtual displacements \( \delta u_1 \)

- \( \delta u_1 \) satisfies DIRICHLET boundary conditions
  \[ \delta u_1 = 0 \quad \forall X_1 \in \Gamma_u \]

- \( \delta u_1 \) satisfies field condition
  \[ \delta \varepsilon_{11} = \delta u_{1,1} \quad \forall X_1 \in \Omega \]

- \( \delta u_1 \) is arbitrary (not equal zero)

- \( \delta u_1 \) is infinite small
• integration over domain Ω (length)

\[ \int_{0}^{L} \delta u_1 \left[ \rho \ddot{u}_1 - \rho b_1 - \sigma_{11,1} \right] dX_1 = 0 \]

\[ \delta u_1 \left[ \sigma_{11} n_1 - t_1^* \right] = 0 \]

• partial integration of stress term

\[ \int_{0}^{L} \delta u_1 \sigma_{11,1} dX_1 = \left[ \delta u_1 \sigma_{11} n_1 \right]_{0}^{L} - \int_{0}^{L} \partial \delta u_1, 1 \sigma_{11} dX_1 \]

• inclusion into integrated balance of momentum

NEUMANN boundary conditions at left and right boundary addition of both terms

\[ \int_{0}^{L} \delta u_1 \left[ \rho \ddot{u}_1 - \rho b_1 \right] dX_1 + \int_{0}^{L} \delta \varepsilon_{11} \sigma_{11} dX_1 \]

\[ - \left[ \delta u_1 \sigma_{11} n_1 \right]_{0}^{L} + \left[ \delta u_1 \sigma_{11} n_1 \right]_{0}^{L} - \left[ \delta u_1 t_1^* \right]_{0}^{L} = 0 \]

• rearrangement

weak form of continuum mechanics - 1d
principle of virtual work

\[
\int_0^L \delta u_1 \rho \ddot{u}_1 \, dX_1 + \int_0^L \delta \varepsilon_{11} \sigma_{11} \, dX_1 \\
= \left[ \delta u_1 \ t_1^* \right]_0^L + \int_0^L \delta u_1 \rho \, b_1 \, dX_1
\]

principle of virtual work

\[
\delta W_{\text{dyn}} + \delta W_{\text{int}} = \delta W_{\text{ext}}
\]

virtual work of inertia forces

\[
\delta W_{\text{dyn}} = \int_0^L \delta u_1 \rho \ddot{u}_1 \, dX_1
\]

internal virtual work

\[
\delta W_{\text{int}} = \int_0^L \delta \varepsilon_{11} \sigma_{11} \, dX_1
\]

external virtual work

\[
\delta W_{\text{ext}} = \left[ \delta u_1 \ t_1^* \right]_0^L + \int_0^L \delta u_1 \rho \, b_1 \, dX_1
\]

principle of virtual work - 1d
displacement $u_1$

\[ \frac{u_1}{\rho b L^2} \]

strain $\varepsilon_{11}$

\[ \frac{\varepsilon_{11}}{\rho b L} \]

model problem - tension bar loaded by $\rho b_1$

$\rho b_1, b_1 = b = \text{constant}$

$E, A, \rho, \Omega = [0, L]$

- principle of virtual work, statics $\ddot{u}_1 = 0$
- **DIRICHLET** and **NEUMANN** boundary conditions $\delta u_1(0) = 0$ and $t_1^*(L) = 0$

\[ \int_0^L \delta \varepsilon_{11} \sigma_{11} \, dX_1 = \int_0^L \delta u_1 \rho b_1 \, dX_1 \]
model problem - tension bar loaded by \( \rho b_1 \)

\[ \rho b_1, b_1 = b = \text{constant} \]

\( E, A, \rho, \Omega = [0, L] \)

- principle of virtual work, statics \( \ddot{u}_1 = 0 \)
- \text{DIRICHLET and NEUMANN} boundary conditions \( \delta u_1(0) = 0 \) and \( t_1^*(L) = 0 \)

\[ \int_0^L \delta \varepsilon_{11} \sigma_{11} \, dX_1 = \int_0^L \delta u_1 \rho b_1 \, dX_1 \]

- linear approximation of solution
  (ansatz of displacement field)

\[ \tilde{u}_1(X_1) = \frac{u_2^1}{L} X_1 \]
displacement ansatz \( \tilde{u}_1 \)

\[ \tilde{u}_1(X_1) = \frac{u_1^2}{L} X_1 \]

constant approximation of normal strain

\[ \tilde{\varepsilon}_{11}(X_1) = \frac{\partial \tilde{u}_1(X_1)}{\partial X_1} = \frac{u_1^2}{L} \]

constant approximation of normal stress

\[ \tilde{\sigma}_{11}(X_1) = E \tilde{\varepsilon}_{11}(X_1) = \frac{E u_1^2}{L} \]

linear approximation of virtual displacement (analogous to real displacement)

\[ \delta \tilde{u}_1(X_1) = \frac{\delta u_1^2}{L} X_1 \]

constant approximation of virtual normal strain

\[ \delta \tilde{\varepsilon}_{11}(X_1) = \frac{\partial \delta \tilde{u}_1(X_1)}{\partial X_1} = \frac{\delta u_1^2}{L} \]
principle of virtual work including approximations

\[
\int_0^L \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 = \int_0^L \delta \tilde{u}_1 \rho b \, dX_1
\]

inclusion of approximations

\[
\int_0^L \frac{\delta u_1^2}{L} \frac{E}{L} u_1^2 \, dx_1 = \int_0^L \frac{\delta u_1^2}{L} X_1 \rho b \, dx_1
\]

integration - real and virtual nodal displacements \( u_1^2 \) and \( \delta u_1^2 \) are independent of \( X_1 \)

\[
\delta u_1^2 \frac{E}{L} u_1^2 = \delta u_1^2 \frac{\rho b L}{2}
\]

\[
\delta u_1^2 \left[ \frac{E}{L} u_1^2 - \frac{\rho b L}{2} \right] = 0
\]

\( \delta u_1^2 \) is arbitrary \( \rightarrow \) term in brackets is zero

\[
\frac{E}{L} u_1^2 - \frac{\rho b L}{2} = 0
\]

\[
u_1^2 = \frac{\rho b}{2 E} L^2
\]

approximated solution is at node 2 identical to analytical solution
postprocessing
calculation of approximated displacement, strain and stress fields

- displacement field

\[ \tilde{u}_1(X_1) = \frac{u_1^2}{L} X_1 = \frac{\rho b L}{2 E} X_1 \]

- strain field - derivative of displacement field

\[ \tilde{\varepsilon}_{11}(X_1) = \tilde{u}_{1,1}(X_1) = \frac{\rho b L}{2 E} \]

- stress field - constitutive law

\[ \tilde{\sigma}_{11}(X_1) = E \tilde{\varepsilon}_{11}(X_1) = \frac{\rho b L}{2} \]

- derivative of stress field

\[ \tilde{\sigma}_{11,1}(X_1) = E \tilde{\varepsilon}_{11,1}(X_1) \]

- residuum (local error) - solution of differential equation

\[ \sigma_{11,1}(X_1) + \rho b = 0 + \rho b = \rho b \]

Local error is not zero → approximated solution is not correct for every \( X_1 \)!
Displacements $u_1$ and $\tilde{u}_1$:

$$\frac{u_1 E}{\rho b L^2} = \frac{1}{2}$$

Strain $\varepsilon_{11}$ and $\tilde{\varepsilon}_{11}$:

$$\frac{\varepsilon_{11} E}{\rho b L} = 1.2$$

Stress $\sigma_{11}$ and $\tilde{\sigma}_{11}$:

$$\frac{\sigma_{11}}{\rho b L} = 1.2$$

Residuum $\sigma_{11,1} + \rho b_1$ and $\tilde{\sigma}_{11,1} + \rho b_1$:

$$\frac{\sigma_{11,1} + \rho b_1}{\rho b} = -1.2$$
check principle of virtual work
- introducing approximations $\delta \tilde{u}_1(X_1)$, $\delta \tilde{u}_{1,1}(X_1)$ and $\tilde{u}_1(X_1)$
- using nodal displacement $u_1^2 = \frac{\rho b}{2E} L^2$

$$\int_0^L \frac{\delta u_1^2}{L} \frac{E u_1^2}{L} dX_1 - \int_0^L \frac{\delta u_1^2}{L} X_1 \rho b dX_1 = \delta u_1^2 \left[ \frac{\rho b}{2} L - \frac{\rho b}{2} L \right] = 0$$

approximated solution fulfills principle of virtual work even if differential equation is not fulfilled!

principle of virtual work is basis for development of numerical methods for mechanics

development of finite element method is based on principle of virtual work

possibilities to improve quality of approximated solution
- subdivision of domain in smaller sub domains using linear ansatz functions ($h$ method)
- using higher order polynomials for approximations ($p$ method)

...
model problem - tension bar loaded by $\rho b_1$

- principle of virtual work

$$\int_0^L \delta \varepsilon_{11} \sigma_{11} \, dX_1 = \int_0^L \delta u_1 \rho \, b_1 \, dX_1$$
displacement ansatz \( \tilde{u}_1 \)

\[ \tilde{u}_1(X_1) = \begin{cases} u_1^2 & \text{domain 1} \\ u_1^3 & \text{domain 2} \end{cases} \]

strain approximation \( \tilde{\varepsilon}_{11} \)

\[ \tilde{\varepsilon}_{11}(X_1) = \begin{cases} \varepsilon_{11}^2 & \text{domain 1} \\ \varepsilon_{11}^3 & \text{domain 2} \end{cases} \]

model problem - tension bar loaded by \( \rho b_1 \)

\[ \rho b_1, b_1 = b = \text{constant} \]

\[ E, A, \rho, \Omega = [0, L] \]

- piecewise linear approximation of displacement field
  - domain 1
    \[ \tilde{u}_1(X_1) = \frac{2 u_1^2}{L} X_1 \]
  - domain 2
    \[ \tilde{u}_1(X_1) = u_1^2 + \frac{2[u_1^3 - u_1^2]}{L} \left[ X_1 - \frac{L}{2} \right] \]
    \[ \tilde{u}_1(X_1) = 2u_1^2 - u_1^3 + \frac{2[u_1^3 - u_1^2]}{L} X_1 \]
- piecewise constant strain approximation
  - domains 1 and 2
  \[
  \tilde{\varepsilon}_{11}(X_1) = \frac{2u_1^2}{L} \quad \tilde{\varepsilon}_{11}(X_1) = \frac{2[u_1^3 - u_1^2]}{L}
  \]

- piecewise constant stress approximation \( \sigma_{11} = E\varepsilon_{11} \)
  - domains 1 and 2
  \[
  \tilde{\sigma}_{11}(X_1) = E \frac{2u_1^2}{L} \quad \tilde{\sigma}_{11}(X_1) = E \frac{2[u_1^3 - u_1^2]}{L}
  \]

- piecewise linear approximation of virtual displacement field
  - domain 1
  \[
  \delta\tilde{u}_1(X_1) = \frac{2\delta u_1^2}{L} X_1
  \]
  - domain 2
  \[
  \delta\tilde{u}_1(X_1) = 2\delta u_1^2 - \delta u_1^3 + \frac{2[\delta u_1^3 - \delta u_1^2]}{L} X_1
  \]

- approximation of virtual strain
  - domains 1 and 2
  \[
  \delta\tilde{\varepsilon}_{11}(X_1) = \frac{2\delta u_1^2}{L} \quad \delta\tilde{\varepsilon}_{11}(X_1) = \frac{2[\delta u_1^3 - \delta u_1^2]}{L}
  \]
principle of virtual work including approximations

\[ \int_0^L \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 = \int_0^L \delta \tilde{u}_1 \rho b_1 \, dX_1 \]

internal virtual work - considering two domains, inducing approximations, integration

\[ \int_0^L \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 = \int_0^{L/2} \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 + \int_{L/2}^L \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 \]

\[ = \int_0^{L/2} \frac{2\delta u_1^2}{L} \frac{2E u_1^2}{L} \, dX_1 + \int_{L/2}^L \frac{2[\delta u_1^3 - \delta u_1^2]}{L} \frac{2E[u_1^3 - u_1^2]}{L} \, dX_1 \]

\[ = \frac{4E\delta u_1^2 u_1^2}{L^2} \int_0^{L/2} \, dX_1 + \frac{4E[\delta u_1^3 - \delta u_1^2][u_1^3 - u_1^2]}{L^2} \int_{L/2}^L \, dX_1 \]

\[ = \frac{2E\delta u_1^2 u_1^2}{L} + \frac{2E[\delta u_1^3 - \delta u_1^2][u_1^3 - u_1^2]}{L} \]

\[ = \delta u_1^2 \frac{2E}{L} u_1^2 + \frac{2E[\delta u_1^3 u_1^3 - \delta u_1^2 u_1 - \delta u_1^2 u_1^3 + \delta u_1^2 u_1^3]}{L} \]
• rearrangement and transformation in matrix format

\[
\begin{align*}
\int_0^L \delta \tilde{\varepsilon}_{11} \tilde{\sigma}_{11} \, dX_1 &= \delta u_1^2 \frac{2E}{L} u_1^2 + \delta u_1^3 \frac{2E}{L} u_1^3 - \delta u_1^3 \frac{2E}{L} u_1^2 - \delta u_1^2 \frac{2E}{L} u_1^3 + \delta u_1^2 \frac{2E}{L} u_1^2 \\
&= \begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix} \cdot \begin{bmatrix} \frac{4E}{L} & -\frac{2E}{L} \\ -\frac{2E}{L} & \frac{2E}{L} \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_1^3 \end{bmatrix} = \delta u \cdot K \, u
\end{align*}
\]

• definition of

- system displacement vector \( u \),
- virtual system displacement vector \( \delta u \)
- and system stiffness matrix \( K \)

\[
\begin{align*}
u &= \begin{bmatrix} u_1^2 \\ u_1^3 \end{bmatrix}, \quad \delta u &= \begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix} \\
K &= \begin{bmatrix} \frac{4E}{L} & -\frac{2E}{L} \\ -\frac{2E}{L} & \frac{2E}{L} \end{bmatrix}
\end{align*}
\]
• external virtual work - considering two domains, inducing approximations, integration

\[
\begin{align*}
\int_0^L \delta \tilde{u}_1 \rho b_1 \, dX_1 &= \int_0^{L/2} \delta \tilde{u}_1 \rho b_1 \, dX_1 + \int_{L/2}^L \delta \tilde{u}_1 \rho b_1 \, dX_1 \\
&= \int_0^{L/2} \frac{2 \delta u_1^2}{L} X_1 \rho b \, dX_1 + \int_{L/2}^L [2 \delta u_1^2 - \delta u_1^3 + \frac{2(\delta u_1^3 - \delta u_1^2)}{L} X_1] \rho b \, dX_1 \\
&= \delta u_1^2 \frac{2 \rho b}{L} \int_0^{L/2} X_1 \, dX_1 + [2 \delta u_1^2 - \delta u_1^3] \rho b \int_{L/2}^L dX_1 \\
&+ [\delta u_1^3 - \delta u_1^2] \frac{2 \rho b}{L} \int_0^{L/2} X_1 \, dX_1 \\
&= \delta u_1^2 \frac{\rho b}{L} \left[[X_1]^2\right]_0^{L/2} + [2 \delta u_1^2 - \delta u_1^3] \rho b \frac{L}{2} \\
&+ [\delta u_1^3 - \delta u_1^2] \frac{\rho b}{L} \left[[X_1]^2\right]_{L/2}^L \\
&= \delta u_1^2 \frac{\rho bL}{4} + [2 \delta u_1^2 - \delta u_1^3] \frac{\rho bL}{2} \\
&+ [\delta u_1^3 - \delta u_1^2] \frac{3 \rho bL}{4}
\end{align*}
\]
• rearrangement and transformation in matrix format

\[
\int_0^L \delta \ddot{u}_1 \rho b_1 \, dX_1 = \delta u_1^2 \left[ \frac{\rho b L}{4} + \rho b L - \frac{3\rho b L}{4} \right] + \delta u_1^3 \left[ -\frac{\rho b L}{2} + \frac{3\rho b L}{4} \right] = \delta u_1^2 \frac{\rho b L}{2} + \delta u_1^3 \frac{\rho b L}{4} = \begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix} \cdot \begin{bmatrix} \frac{\rho b L}{2} \\ \frac{\rho b L}{4} \end{bmatrix} = \delta \mathbf{u} \cdot \mathbf{r}
\]

• definition of
  ○ virtual system displacement vector \( \delta \mathbf{u} \)
  ○ and system load vector \( \mathbf{r} \)

\[
\delta \mathbf{u} = \begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} \frac{\rho b L}{2} \\ \frac{\rho b L}{4} \end{bmatrix}
\]

• principle of virtual displacements including approximations

\[
\begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix} \cdot \begin{bmatrix} \frac{4E}{L} & -\frac{2E}{L} \\ -\frac{2E}{L} & \frac{2E}{L} \end{bmatrix} \begin{bmatrix} u_1^2 \\ u_1^3 \end{bmatrix} = \begin{bmatrix} \delta u_1^2 \\ \delta u_1^3 \end{bmatrix} \cdot \begin{bmatrix} \frac{\rho b L}{2} \\ \frac{\rho b L}{4} \end{bmatrix}
\]

---

tension bar - finite element solution
- rearrangement

\[
\begin{bmatrix}
\delta u^2 \\
\delta u^3
\end{bmatrix}
\cdot
\begin{bmatrix}
\frac{4E}{L} & -\frac{2E}{L} \\
-\frac{2E}{L} & \frac{2E}{L}
\end{bmatrix}
\begin{bmatrix}
u^2 \\
u^3
\end{bmatrix}
- \begin{bmatrix}
\frac{\rho b L}{2} \\
\frac{\rho b L}{4}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

- for arbitrary virtual displacements \(\delta u^2\) and \(\delta u^3\)
(or for arbitrary virtual system displacement vector \(\delta u\))

\[
\begin{bmatrix}
\frac{4E}{L} & -\frac{2E}{L} \\
-\frac{2E}{L} & \frac{2E}{L}
\end{bmatrix}
\begin{bmatrix}
u^2 \\
u^3
\end{bmatrix}
= \begin{bmatrix}
\frac{\rho b L}{2} \\
\frac{\rho b L}{4}
\end{bmatrix}
\]

- system stiffness relation

\[K \ u = r\]

- solution of linear system of equations \(u^2\) und \(u^3\)

\[
\begin{bmatrix}
\frac{4E}{L} & -\frac{2E}{L} \\
0 & \frac{2E}{L}
\end{bmatrix}
\begin{bmatrix}
u^2 \\
u^3
\end{bmatrix}
= \begin{bmatrix}
\frac{\rho b L}{2} \\
\rho b L
\end{bmatrix},
\]

\[u^3 = \frac{\rho b}{2E} \ L^2, \quad u^1 = \frac{\rho b}{2E} \ L, \quad u = \frac{L}{4E} \left[ \frac{\rho b L}{2} + \frac{2E}{L} \frac{\rho b}{2E} \ L^2 \right] = \frac{3\rho b}{8E} \ L^2\]

---

tension bar - finite element solution
postprocessing - calculation of approximated displacement, strain and stress fields

- Verschiebungsfeld
  - domain 1
    \[ \tilde{u}_1(X_1) = \frac{2}{L} u_1^2 \quad X_1 = \frac{3\rho b L}{4E} \quad X_1 \]
  - domain 2
    \[ \tilde{u}_1(X_1) = 2u_1^2 - u_1^3 - \frac{2[u_1^2 - u_1^3]}{L} \quad X_1 = \frac{\rho b}{4E}L^2 + \frac{\rho b}{4E}L \quad X_1 = \frac{\rho b}{4E}L[L + X_1] \]

- strain field - derivative of displacement fields in domains 1 and 2
  \[ \tilde{\varepsilon}_{11}(X_1) = \frac{3\rho b L}{4E} \quad \tilde{\varepsilon}_{11}(X_1) = \frac{\rho b}{4E}L \]

- stress field an derivative of stress field in domains 1 and 2
  \[ \tilde{\sigma}_{11}(X_1) = E \tilde{\varepsilon}_{11}(X_1) = \frac{\rho b L}{2} \quad \tilde{\sigma}_{11,1}(X_1) = E \tilde{\varepsilon}_{11,1}(X_1) \]

- residuum (local error) - solution of differential equation
  \[ \sigma_{11,1}(X_1) + \rho b_1 = 0 + \rho b = \rho b \]
  local error is not zero \(\rightarrow\) approximated solution is not correct for every \(X_1\)!
displacements $u_1$ and $\tilde{u}_1$

\[
\frac{u_1 E}{\rho b L^2} = \frac{3}{8} \frac{1}{2}
\]

strains $\varepsilon_{11}$ and $\tilde{\varepsilon}_{11}$

\[
\frac{\varepsilon_{11} E}{\rho b L} = 1.2
\]

stresses $\sigma_{11}$ and $\tilde{\sigma}_{11}$

\[
\frac{\sigma_{11}}{\rho b L} = 1.2
\]

residuals $\sigma_{11,1} + \rho b_1$ and $\tilde{\sigma}_{11,1} + \rho b_1$

\[
\frac{\sigma_{11,1} + \rho b_1}{\rho b} = -1.2
\]
\[
\text{displacement } u_1 \\
\begin{array}{l}
\text{analytical solution} \\
\text{fe solution}
\end{array}
\]

\[
\text{strain } \varepsilon_{11} \\
\sigma_{11} = E \varepsilon_{11}
\]

tension bar - convergence of fe solution
- subdivision of domain $\Omega$ in subdomains or elements $\Omega^e$ with $e \in [1, 2]$
- generation of elements with linear approximations of displacement field
- derivation of approximations for stress, strain, virtual displacement and virtual strain fields
- application of principle of virtual displacements within domain $\Omega$ using approximations ($\delta \tilde{W}_{\text{dyn}} + \delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{ext}}$)
- rearrangement: moving virtual and real nodal displacements ($\delta u^2_1$, $\delta u^3_1$, $u^2_1$ and $u^3_1$) outside integrals ($\delta u^i_1$ to left hand side, $u^i_1$ to right hand side)
- integration of virtual works of hole domain using summation over elements
- transformation to matrix format, application for arbitrary virtual displacements
- solution of system stiffness relation
- postprocessing calculating approximated displacement, strain and stress fields

main aspect of present finite element solution
postive aspects of present finite element solution

- approximative solution of elasticity can be found by using simple mathematics (statement is also valid for differential equations which can not be solved analytically)
- approximated numerical solution converges to the analytical solution, using an increased number of elements ($h$ method) or an increased polynomial degree ($p$ method)
- values interesting for engineers (strain and stress) can be calculated by post processing procedure

negative aspects of present finite element solution

- only approximation of solution which should be evaluated by engineer
- ansatz functions have been hardly developed, no systematic formulation included
- hand made rearrangement of approximated weak forms are individual for every problem
- procedure is not well suited for computational implementation
optimization of finite element concept
computer oriented formulation

- strict separation of structural and element levels
- systematic design of displacement approximation on element level
- generation of weak form on the element level
\[ \delta \tilde{W}_e^{\text{dyn}} + \delta \tilde{W}_e^{\text{int}} = \delta \tilde{W}_e^{\text{ext}} \]
- systematic integration of virtual work terms and derivation of element matrices and vectors
- generation of structural equation of motion (structural stiffness relation) by assembling of elements
- postprocessing on element level
phenomenological levels of the finite element method

- structural level
- component level
- material point level

\[ \mathbf{K} \mathbf{u} = \mathbf{r} \]
\[ \mathbf{k}^e \mathbf{u}^e = \mathbf{r}^e \]
\[ N_1 = EA \varepsilon_{11} \]

reality

discretized model
phenomenological levels of the finite element method

- **Composite bridge**
- **Piece of road**
- **Concrete**

- **Structural level**
  \[ K \mathbf{u} = \mathbf{r} \]
  - System of equations

- **Component level**
  \[ k^e \mathbf{u}^e = r^e \]
  - Finite elements

- **Material point level**
  \[ \sigma = \mathbf{C} : \varepsilon \]
  - Constitutive law

Reality \[\rightarrow\] Discretized model
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<th>System level (structure)</th>
<th>Element level (finite element)</th>
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finite element method concept
analytical solution

system, structure or domain $\Omega$

concept

- subdivision of structure in finite elements $e$
- definition of structural nodal displacements $u^j_1$ as sampling points of piecewise linear displacement approximations $\tilde{u}_1$
- definition of elements, element nodes and element nodal displacements $u^{ei}_1$
- design of displacement approximation within elements $e$
subdivision in elements, structural nodal displacements and displacement approximation

$$u^1_1 = 0$$

associated elemental nodal displacements and displacement approximation

**Design of displacement approximation**
design of displacement approximation
The document describes the concept of substitution of distributed loads by nodal forces yielding the same virtual work as the original distributed load. It explains the realization of this concept through the evaluation of integrals for element load vectors or 'tensors' in 1D linear finite element method. The equations for the element load vectors $\mathbf{r}^e$ are given for constant and linear $b_1$ elements, as well as for a quadratic element load vector.

For a constant $b_1$ linear element:

$$\mathbf{r}^e = \begin{bmatrix} r_{11}^{e1} \\ r_{12}^{e2} \end{bmatrix} = \frac{\rho b_1 L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For a linear $b_1$ linear element:

$$\mathbf{r}^e = \frac{\rho L}{6} \begin{bmatrix} 2b_1^{e1} + b_1^{e2} \\ b_1^{e1} + 2b_1^{e2} \end{bmatrix}$$

For a constant $b_1$ quadratic element:

$$\mathbf{r}^e = \begin{bmatrix} r_{11}^{e1} \\ r_{11}^{e2} \\ r_{11}^{e3} \end{bmatrix} = \frac{\rho b_1 L}{6} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
• element stiffness 'tensor'

\[ k^{eij} = \int_{-1}^{1} N^i_1(\xi_1) E N^j_1(\xi_1) \frac{L}{2} \, d\xi_1 \]

• element stiffness matrix for \( p = 1 \)

\[ k^e = \begin{bmatrix} k^{e11} & k^{e12} \\ k^{e21} & k^{e22} \end{bmatrix} = \frac{E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

• element load 'tensor'

\[ r^{ei} = \int_{-1}^{1} N^i(\xi_1) \rho \, b_1(\xi_1) \frac{L}{2} \, d\xi_1 \]

• element load vector

\[ r^e = \begin{bmatrix} r^{e1} \\ r^{e2} \end{bmatrix} \]

• element mass 'tensor'

\[ m^{eij} = \int_{-1}^{1} N^i(\xi_1) \rho \, N^j_1(\xi_1) \frac{L}{2} \, d\xi_1 \]

• element mass matrix for \( p = 1 \)

\[ m^e = \begin{bmatrix} m^{e11} & m^{e12} \\ m^{e21} & m^{e22} \end{bmatrix} = \frac{\rho L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

• element load vector for \( p = 1 \) and constant \( b_1 \)

\[ r^e = \begin{bmatrix} r^{e1} \\ r^{e2} \end{bmatrix} = \frac{\rho \, b_1 \, L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]

• element load vector for \( p = 1 \) and linear \( b_1 \)

\[ r^e = \begin{bmatrix} r^{e1} \\ r^{e2} \end{bmatrix} = \frac{\rho \, L}{6} \begin{bmatrix} 2b_1^{e1} + b_1^{e2} \\ b_1^{e1} + 2b_1^{e2} \end{bmatrix} \]
• approximated weak form of element $e$

$$\delta \tilde{W}^e_{\text{dyn}} + \delta \tilde{W}^e_{\text{int}} = \delta \tilde{W}^e_{\text{ext}}$$

• approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

• example - graphical interpretation of $\delta u^e \cdot k^e u^e$ using FALK matrix scheme
- approximated weak form of element $e$

$$\delta \tilde{W}_\text{dyn}^e + \delta \tilde{W}_\text{int}^e = \delta \tilde{W}_\text{ext}^e$$

- approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

- example - graphical interpretation of $\delta u^e \cdot k^e u^e$ using FALK matrix scheme
• approximated weak form of element $e$

$$\delta \tilde{W}^e_{\text{dyn}} + \delta \tilde{W}^e_{\text{int}} = \delta \tilde{W}^e_{\text{ext}}$$

• approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

• example - graphical interpretation of $\delta u^e \cdot k^e u^e$ using FALK matrix scheme

approximation of weak form
approximated weak form of element $e$

$$\delta \tilde{W}_\text{dyn}^e + \delta \tilde{W}_\text{int}^e = \delta \tilde{W}_\text{ext}^e$$

approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

element - graphical interpretation of $\delta u^e \cdot k^e u^e$ using FALK matrix scheme

- rows of $k^e$ correspond to virtual displacements $\delta u_1^{e_i}$
- columns of $k^e$ correspond to real displacements $u_1^{e_j}$
• approximated weak form of element $e$

$$\delta \tilde{W}^e_{\text{dyn}} + \delta \tilde{W}^e_{\text{int}} = \delta \tilde{W}^e_{\text{ext}}$$

• approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

• example - graphical interpretation of $\delta u^e \cdot k^e u^e$ using FALK matrix scheme

- rows of $k^e$ correspond to virtual displacements $\delta u^e_i$
- columns of $k^e$ correspond to real displacements $u^e_j$
• approximated weak form of element $e$

$$\delta\tilde{W}_e^{\text{dyn}} + \delta\tilde{W}_e^{\text{int}} = \delta\tilde{W}_e^{\text{ext}}$$

• approximated weak form of element $e$ using element matrices and vectors

$$\delta u^e \cdot m^e \ddot{u}^e + \delta u^e \cdot k^e u^e = \delta u^e \cdot r^e$$

• approximated weak form of system / structure

$$\delta\tilde{W}_\text{dyn} + \delta\tilde{W}_\text{int} = \delta\tilde{W}_\text{ext}$$

• approximated weak form of system with
  ○ virtual structural displacement vector $\delta u$ and structural displacement vector $u$
  ○ structural acceleration vector $\ddot{u}$ and structural mass matrix $M$
  ○ structural stiffness matrix $K$ and structural load vector $r$

$$\delta u \cdot M \ddot{u} + \delta u \cdot K u = \delta u \cdot r$$

• relation between element and structural quantities? basis is given by summation of element virtual works

$$\delta\tilde{W}_{\text{int,dyn,ext}} = \sum_{e=1}^{NE} \delta\tilde{W}_{\text{int,dyn,ext}}^e$$

---

assembly of elements to structure
finite element system with element nodal displacements

\[
\delta \tilde{W}^e_{\text{int}}
\]

\[
\begin{pmatrix}
\delta u^e_1 \\
\delta u^e_2
\end{pmatrix}
\begin{pmatrix}
k^{e11} & k^{e12} \\
k^{e21} & k^{e22}
\end{pmatrix}
\begin{pmatrix}
u^e_1 \\
u^e_2
\end{pmatrix}
\]

addition of element virtual works
Finite element system with element nodal displacements

\[
\delta \tilde{W}_e = k_e^{11} \delta u_1^e + k_e^{12} \delta u_2^e + k_e^{21} \delta u_1^e + k_e^{22} \delta u_2^e
\]

\[
\delta \tilde{W}_{\text{int}} = u_1^a \delta u_1^a + u_2^a \delta u_2^a + u_1^e \delta u_1^e + u_2^e \delta u_2^e
\]

Addition of element virtual works
finite element system with element nodal displacements

\[
\begin{align*}
\delta \tilde{W}^a_{\text{int}} &= \delta u_1^a \begin{bmatrix} u_1^a & u_1^a \\ k_1^{a1} & k_1^{a2} \end{bmatrix} \\
\delta \tilde{W}^e_{\text{int}} &= \delta u_1^e \begin{bmatrix} u_1^e & u_1^e \\ k_1^{e1} & k_1^{e2} \end{bmatrix} \\
\delta \tilde{W}^c_{\text{int}} &= \delta u_1^c \begin{bmatrix} u_1^c & u_1^c \\ k_1^{c1} & k_1^{c2} \end{bmatrix}
\end{align*}
\]
\[
\delta\tilde{W}_{\text{int}} = \delta\tilde{W}_{\text{int}}^a + \delta\tilde{W}_{\text{int}}^e + \delta\tilde{W}_{\text{int}}^c
\]

finite element system with element nodal displacements
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

relation between system and element nodal displacements

addition of element virtual works
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}^a_{\text{int}} + \delta \tilde{W}^e_{\text{int}} + \delta \tilde{W}^c_{\text{int}}
\]

Relation between system and element nodal displacements

Addition of element virtual works
\[ \delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c \]

\[
\begin{array}{|c|c|}
\hline
0 & u_1^2 \\
\hline
u_1^{a1} & u_1^{a2} \\
\hline
\delta u_1^2 & \delta u_1^{a2} \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
u_1^2 & u_1^3 \\
\hline
u_1^{e1} & u_1^{e2} \\
\hline
\delta u_1^2 & \delta u_1^{e2} \\
\hline
k_1^{e11} & k_1^{e12} \\
\hline
k_1^{e21} & k_1^{e22} \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
u_1^{c1} & u_1^{c2} \\
\hline
\delta u_1^{c1} & \delta u_1^{c2} \\
\hline
k_c^{11} & k_c^{12} \\
\hline
k_c^{21} & k_c^{22} \\
\hline
\end{array}
\]

**relation between system and element nodal displacements**

**addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_a^e + \delta \tilde{W}_e^e + \delta \tilde{W}_c^e
\]

Relation between system and element nodal displacements

**Addition of element virtual works**

linear computational structural mechanics

- 2 1d finite element method
- 2.8 assembly and solution
The generation of structural stiffness matrix by assembly is expressed as the addition of element virtual works to the internal work:

\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

where

- \( \delta \tilde{W}_{\text{int}}^a \) represents the virtual work of the assembly process involving the assembly parameters.
- \( \delta \tilde{W}_{\text{int}}^e \) represents the virtual work of the elements.
- \( \delta \tilde{W}_{\text{int}}^c \) represents the virtual work of the connections.

The matrices associated with these virtual works are:

- For \( \delta \tilde{W}_{\text{int}}^a \):
  - 

- For \( \delta \tilde{W}_{\text{int}}^e \):
  - 

- For \( \delta \tilde{W}_{\text{int}}^c \):
  - 

These matrices are assembled to form the overall stiffness matrix, as illustrated in the diagram.
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

<table>
<thead>
<tr>
<th>0</th>
<th>(u_2^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_1^a)</td>
<td>(u_1^a)</td>
</tr>
<tr>
<td>(k_1^{a1})</td>
<td>(k_1^{a2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\delta u_1^2)</th>
<th>(\delta u_1^3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1^{a1})</td>
<td>(k_1^{a2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\delta u_1^4)</th>
<th>(\delta u_1^e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1^{e1})</td>
<td>(k_1^{e2})</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(\delta u_3^1)</th>
<th>(\delta u_3^c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k_1^{c1})</td>
<td>(k_1^{c2})</td>
</tr>
</tbody>
</table>

**generation of structural stiffness matrix by assembly**

**addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

<table>
<thead>
<tr>
<th></th>
<th>(u_1^2)</th>
<th>(u_1^3)</th>
<th>(u_1^4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(u_1^2)</td>
<td>(u_1^3)</td>
<td>(u_1^4)</td>
</tr>
<tr>
<td>(\delta u_1^2)</td>
<td>(\delta u_1^3)</td>
<td>(k_{e11})</td>
<td>(k_{e12})</td>
</tr>
<tr>
<td>(\delta u_1^3)</td>
<td>(\delta u_1^4)</td>
<td>(k_{e21})</td>
<td>(k_{e22})</td>
</tr>
</tbody>
</table>

**generation of structural stiffness matrix by assembly**

<table>
<thead>
<tr>
<th></th>
<th>(0)</th>
<th>(u_1^2)</th>
<th>(u_1^3)</th>
<th>(u_1^4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(u_1^2)</td>
<td>(u_1^3)</td>
<td>(u_1^4)</td>
<td>(u_1^4)</td>
</tr>
<tr>
<td>(\delta u_1^2)</td>
<td>(\delta u_1^3)</td>
<td>(k_{c11})</td>
<td>(k_{c12})</td>
<td></td>
</tr>
<tr>
<td>(\delta u_1^3)</td>
<td>(\delta u_1^4)</td>
<td>(k_{c21})</td>
<td>(k_{c22})</td>
<td></td>
</tr>
</tbody>
</table>

**addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}^a_{\text{int}} + \delta \tilde{W}^e_{\text{int}} + \delta \tilde{W}^c_{\text{int}}
\]

\[
\begin{array}{|c|c|}
\hline
\delta u_1^2 & 0 \\
\hline
\delta u_1^3 & \delta u_1^{a1} \\
\hline
\delta u_1^4 & \delta u_1^{a2} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
0 & 0 \\
\hline
\delta u_1^1 & k_1^{a11} \\
\hline
\delta u_1^1 & k_1^{a12} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
0 & \delta u_1^1 \\
\hline
\delta u_1^2 & \delta u_1^{e1} \\
\hline
\delta u_1^2 & \delta u_1^{e2} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
u_1^2 & u_1^3 \\
\hline
u_1^3 & u_1^4 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\delta u_1^1 & k_1^{e11} \\
\hline
\delta u_1^1 & k_1^{e12} \\
\hline
\delta u_1^4 & k_1^{c11} \\
\hline
\delta u_1^4 & k_1^{c12} \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
k_1^{a21} & k_1^{a22} \\
\hline
k_1^{e21} & k_1^{e22} \\
\hline
k_1^{c21} & k_1^{c22} \\
\hline
\end{array}
\]

**Generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} =
\begin{array}{|c|c|c|c|}
\hline
0 & u_1^2 & u_1^3 & u_1^4 \\
\hline
0 & K_1^{11} & K_1^{12} & K_1^{13} & K_1^{14} \\
\hline
0 & K_1^{21} & K_1^{22} & K_1^{23} & K_1^{24} \\
\hline
0 & K_1^{31} & K_1^{32} & K_1^{33} & K_1^{34} \\
\hline
0 & K_1^{41} & K_1^{42} & K_1^{43} & K_1^{44} \\
\hline
\end{array}
\]

**Addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

<table>
<thead>
<tr>
<th>( \delta \tilde{W}_{\text{int}}^a )</th>
<th>( \delta \tilde{W}_{\text{int}}^e )</th>
<th>( \delta \tilde{W}_{\text{int}}^c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>( u^2_1 )</td>
<td>( u^3_1 )</td>
</tr>
<tr>
<td>( u^a_1 )</td>
<td>( u^a_2 )</td>
<td>( u^e_1 )</td>
</tr>
<tr>
<td>( \delta u^2_1 )</td>
<td>( k^{a11} )</td>
<td>( k^{a12} )</td>
</tr>
<tr>
<td>( \delta u^3_1 )</td>
<td>( k^{a21} )</td>
<td>( k^{a22} )</td>
</tr>
</tbody>
</table>

**Generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} = \begin{bmatrix}
0 & u^2_1 & u^3_1 & u^4_1 \\
\delta u^2_1 & K^{22} & K^{23} & K^{24} \\
\delta u^3_1 & K^{32} & K^{33} & K^{34} \\
\delta u^4_1 & K^{42} & K^{43} & K^{44}
\end{bmatrix}
\]

**Addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

\[
\begin{array}{c|c|c}
0 & u_2^1 & 0 \\
\hline
u_1^{a1} & u_1^{a2} & \delta u_1^2 \\
\hline
\delta u_1^2 & k_a^{21} & k_a^{22} \\
\end{array}
\quad
\begin{array}{c|c|c}
\delta u_1^1 & \delta u_1^e & \delta u_1^3 \\
\hline
\delta u_1^2 & k_e^{11} & k_e^{12} \\
\hline
\delta u_1^3 & k_e^{21} & k_e^{22} \\
\end{array}
\quad
\begin{array}{c|c|c}
\delta u_1^3 & \delta u_1^c & \delta u_1^4 \\
\hline
\delta u_1^2 & k_c^{11} & k_c^{12} \\
\hline
\delta u_1^3 & k_c^{21} & k_c^{22} \\
\end{array}
\]

**generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} = \begin{array}{c|c|c|c}
& u_2^1 & u_3^1 & u_4^1 \\
\hline
\delta u_1^2 & k_a^{22} & k_e^{11} & \delta u_1^3 \\
\delta u_1^3 & k_e^{21} & \delta u_1^4 \\
\delta u_1^4 & \delta u_1^c & \delta u_1^2 \\
\end{array}
\]

**addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

| \delta \tilde{W}_{\text{int}} | \delta \tilde{W}_{\text{int}}^a | + | \delta \tilde{W}_{\text{int}}^e | + | \delta \tilde{W}_{\text{int}}^c |
|------------------|------------------|------------------|------------------|
| \alpha^1 \beta^1 | \alpha^2 \beta^2 | \alpha^3 \beta^3 | \alpha^4 \beta^4 |
| \delta u_1^2 \delta u_1^3 \delta u_1^4 | \delta u_1^2 \delta u_1^3 \delta u_1^4 | \delta u_1^2 \delta u_1^3 \delta u_1^4 | \delta u_1^2 \delta u_1^3 \delta u_1^4 |
| 0 | \alpha^1 \beta^1 | \alpha^2 \beta^2 | \alpha^3 \beta^3 | \alpha^4 \beta^4 |
| \delta u_1^2 | \delta u_1^3 | \delta u_1^4 | \delta u_1^5 | \delta u_1^6 |
| \delta u_1^2 | \delta u_1^3 | \delta u_1^4 | \delta u_1^5 | \delta u_1^6 |
| \delta u_1^2 | \delta u_1^3 | \delta u_1^4 | \delta u_1^5 | \delta u_1^6 |

**Generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} = \begin{pmatrix}
0 & u_1^2 \\
\delta u_1^2 & \alpha^2 \beta^2 \\
\delta u_1^3 & \delta u_1^3 \\
\delta u_1^4 & \delta u_1^4 \\
\end{pmatrix} + \begin{pmatrix}
0 & u_1^3 \\
0 & u_1^3 \\
0 & u_1^3 \\
0 & u_1^3 \\
\end{pmatrix} + \begin{pmatrix}
0 & u_1^4 \\
0 & u_1^4 \\
0 & u_1^4 \\
0 & u_1^4 \\
\end{pmatrix}
\]
\[ \delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c \]

**Generation of structural stiffness matrix by assembly**

\[ \delta \tilde{W}_{\text{int}} = \left[ \begin{array}{ccc} u_2^1 & u_3^1 & u_4^1 \\ \delta u_1^2 & \delta u_3^1 & \delta u_4^1 \\ \delta u_2^2 & \delta u_3^2 & \delta u_4^2 \end{array} \right] \]

**Addition of element virtual works**

- \( k_{a22} \) and \( k_{e22} \)
- \( k_{e11}, k_{e12}, k_{c11}, k_{c12}, k_{c21}, k_{c22} \)
\[
s_{\text{int}} = s_{\text{int}}^a + s_{\text{int}}^e + s_{\text{int}}^c
\]

**Generation of Structural Stiffness Matrix by Assembly**

\[
\delta \tilde{W}_{\text{int}} = \begin{array}{ccc}
0 & u_2^1 & u_2^2 \\
\delta u_1^1 & k_a^{11} & k_a^{12} \\
\delta u_1^2 & k_a^{21} & k_a^{22}
\end{array} + \begin{array}{ccc}
\delta u_1^2 & \delta u_1^e & \delta u_1^c \\
k_e^{11} & k_e^{12} & k_c^{11} \\
k_e^{21} & k_e^{22} & k_c^{21}
\end{array} + \begin{array}{ccc}
\delta u_1^3 & \delta u_1^c & \delta u_1^c \\
k_c^{11} & k_c^{12} & k_c^{21} \\
k_c^{21} & k_c^{22}
\end{array}
\]

**Addition of Element Virtual Works**
\[ \delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c \]

<table>
<thead>
<tr>
<th></th>
<th>( \delta u_1^1 )</th>
<th>( \delta u_1^2 )</th>
<th>( \delta u_1^3 )</th>
<th>( \delta u_1^4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta u_2^2 )</td>
<td>0</td>
<td>( u_1^2 )</td>
<td>( u_1^3 )</td>
<td>( u_1^4 )</td>
</tr>
<tr>
<td>( \delta u_2^2 )</td>
<td>( k_{a11} )</td>
<td>( k_{a12} )</td>
<td>( k_{e11} )</td>
<td>( k_{e12} )</td>
</tr>
<tr>
<td>( \delta u_2^2 )</td>
<td>( k_{a21} )</td>
<td>( k_{a22} )</td>
<td>( k_{e21} )</td>
<td>( k_{e22} )</td>
</tr>
</tbody>
</table>

**generation of structural stiffness matrix by assembly**

\[ \delta \tilde{W}_{\text{int}} = \]

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( \delta u_1^1 )</th>
<th>( \delta u_1^2 )</th>
<th>( \delta u_1^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \delta u_2^2 )</td>
<td></td>
<td>( k_{a22}k_{e11} )</td>
<td>( k_{e12} )</td>
<td>0</td>
</tr>
<tr>
<td>( \delta u_2^3 )</td>
<td>( k_{e21} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \delta u_2^4 )</td>
<td>( k_{e21} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


**addition of element virtual works**
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

\[
\begin{array}{cccc}
0 & u_2^2 &  & \\
\delta u_1^a & k^{a11} & k^{a12} & \\
\delta u_2^a & k^{a21} & k^{a22} & \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & u_2^1 & u_3^1 & u_4^1 \\
\delta u_1^e & k^{e11} & k^{e12} & \\
\delta u_2^e & k^{e21} & k^{e22} & \\
\end{array}
\]

\[
\begin{array}{cccc}
u_3^3 & u_1^c &  & \\
\delta u_1^c & k^{c11} & k^{c12} & \\
\delta u_2^c & k^{c21} & k^{c22} & \\
\end{array}
\]

**generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} =
\]

\[
\begin{array}{cccc}
0 & u_2^1 & u_3^1 & u_4^1 \\
\delta u_1^a & k^{a22}+k^{e11} & k^{e12} & 0 \\
\delta u_2^a & k^{e21} & k^{e22} & k^{c11} \\
\delta u_1^c & k^{c21} & k^{c22} & \\
\end{array}
\]

**addition of element virtual works**

---

d.kuhl, wes.online, university of kassel

linear computational structural mechanics • 2.1d finite element method • 2.8 assembly and solution
\[
\delta\tilde{W}_{\text{int}} = \delta\tilde{W}_{\text{int}}^a + \delta\tilde{W}_{\text{int}}^e + \delta\tilde{W}_{\text{int}}^c
\]

| \(0\) | \(u_2^1\) |
| \(u_1^a\) | \(u_1^a\) |
| \(0\) | \(\delta u_1^a\) |
| \(k_a^{11}\) | \(k_a^{12}\) |
| \(\delta u_1^2\) | \(k_a^{21}\) | \(k_a^{22}\) |

| \(u_1^a\) | \(u_1^2\) |
| \(u_1^e\) | \(u_1^e\) |
| \(\delta u_1^2\) | \(\delta u_1^e\) |
| \(k_e^{11}\) | \(k_e^{12}\) |
| \(\delta u_1^3\) | \(k_e^{21}\) | \(k_e^{22}\) |

| \(u_1^c\) | \(u_1^1\) |
| \(u_1^c\) | \(u_1^c\) |
| \(\delta u_1^3\) | \(\delta u_1^c\) |
| \(k_c^{11}\) | \(k_c^{12}\) |
| \(\delta u_1^4\) | \(k_c^{21}\) | \(k_c^{22}\) |

---

generation of structural stiffness matrix by assembly

\[
\delta\tilde{W}_{\text{int}} =
\begin{array}{c|c|c|c}
\delta u_1^2 & u_1^2 & u_1^3 & u_1^4 \\
\hline
k_a^{22} + k_e^{11} & k_e^{12} & 0 & \\
k_e^{21} & k_e^{22} + k_c^{11} & k_c^{12} & \\
0 & k_c^{21} & k_c^{22} & \\
\end{array}
\]

---

addition of element virtual works
\[
\delta \tilde{W}_{\text{int}} = \delta \tilde{W}_{\text{int}}^a + \delta \tilde{W}_{\text{int}}^e + \delta \tilde{W}_{\text{int}}^c
\]

\[
\begin{array}{c|c|c}
0 & u_2^1 & u_1^2 \\
\hline
u_1^{a1} & u_1^{a2} & 0 \\
\hline
\delta u_1^2 & \delta u_1^{a2} & k_1^{a12} \\
\hline
\delta u_1^2 & \delta u_1^{a1} & k_1^{a11} \\
\end{array}
\begin{array}{c|c|c}
0 & u_3^1 & u_1^3 \\
\hline
u_1^{e1} & u_1^{e2} & 0 \\
\hline
\delta u_1^3 & \delta u_1^{e2} & k_1^{e12} \\
\hline
\delta u_1^3 & \delta u_1^{e1} & k_1^{e11} \\
\end{array}
\begin{array}{c|c|c}
0 & u_4^1 & u_1^4 \\
\hline
u_1^{c1} & u_1^{c2} & 0 \\
\hline
\delta u_1^4 & \delta u_1^{c2} & k_1^{c12} \\
\hline
\delta u_1^4 & \delta u_1^{c1} & k_1^{c11} \\
\end{array}
\]

**generation of structural stiffness matrix by assembly**

\[
\delta \tilde{W}_{\text{int}} =
\]

\[
\begin{array}{c|c|c|c}
\delta u_1^2 & k_1^{a22} + k_1^{e12} & k_1^{e12} & 0 \\
\hline
k_1^{e21} & k_1^{e22} + k_1^{c11} & k_1^{c12} & 0 \\
\hline
0 & k_1^{c21} & k_1^{c22} & 0 \\
\end{array}
\]

\[
\delta \tilde{W}_{\text{int}} = \delta u \cdot K \cdot u
\]

**addition of element virtual works**
counting of structural degrees of freedom \rightarrow \text{equation number}

storage of assembly information

- structural nodes and displacements (id)
  - vertical: structural node numbers
  - horizontal: spatial direction (here just 1)
  - array: equation numbers dof’s
- element and structural nodes (it)
  - vertical: element node numbers
  - horizontal: element numbers
  - array: structural node number

\text{computer-oriented assembly procedure}
counting of structural degrees of freedom → equation number

storage of assembly information

- spatial direction
- element number
- equation number
- structural node number

read-out assembly information

- input: element no. \( e \) and node no. \( i \)
- equation number (position in \( K, M, r \))
- output: structural node

computer-oriented assembly procedure
$NE = 4, p = 1, NEQ = 4$

- **Equation number**
- **Structural node number**
- **Element number**

$NE = 8, p = 1, NEQ = 8$

- **Equation number**
- **Structural node number**
- **Element number**

$NE = 4, p = 2, NEQ = 8$

- **Equation number**
- **Structural node number**
- **Element number**

**Computer-oriented assembly procedure**
• assembly is element wise addition of stiffness 'tensors' into the structural stiffness matrix (at the correct position)

• assembly of structural mass matrix and structural load vector is analogous to the element stiffness matrix

• approximated weak form of structure / system

\[ \delta u \cdot M \ddot{u} + \delta u \cdot K u = \delta u \cdot r \]

• reformulation

\[ \delta u \cdot [M \ddot{u} + K u - r] = 0 \]

• fundamental lemma of variational calculus (arbitrary virtual displacements)

\[ M \ddot{u} + K u = r \]

○ ordinary second order (time) differential equation

○ eigen value analysis and time integration

• time independent case - structural stiffness relation

\[ K u = r \]

○ linear system of equations for solution of structural displacement vector \( u \)
motivation - normal loading tower

dead load $p_1$

cross section $A$

tension bar - finite element solution ii
tension bar loaded by $\rho b_1$

\[ \begin{align*}
\rho b_1, b_1 = b = \text{constant} \\
E, \rho, A
\end{align*} \]

\[ L \]

- element stiffness matrix with element length $L^a = L$

\[ k^a = \frac{E}{L^a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]

- element load vector for constant $b_1 = b$

\[ r^a = \frac{\rho b L^a}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
tension bar loaded by $\rho b_1$

$\rho b_1, b_1 = b = \text{constant}$

$E, \rho, A$

$L$

- **element stiffness matrix with element length $L^a = L$**

$$k^a = \frac{E}{L^a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- **element load vector for constant $b_1 = b$**

$$r^a = \frac{\rho b L^a}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- **structural stiffness relation $Ku = r$**

\[
\begin{bmatrix}
\frac{E}{L} & -\frac{E}{L} \\
-\frac{E}{L} & \frac{E}{L}
\end{bmatrix}
\begin{bmatrix}
u_1^1 \\
u_1^2
\end{bmatrix}
= \frac{\rho b L}{2}
\]

fe discretization

system / structural level

$u_1^1 = 0$

$u_1^a$

$u_1^a$

$u_1^a$

$u_1^a$

element level

one element

tension bar - finite element solution ii
tension bar loaded by $\rho b_1$

\[ \rho b_1, b_1 = b = \text{constant} \]

\[ E, \rho, A \]

\[ L \]

- element stiffness matrix with element length $L^a = L$

\[
k^a = \frac{E}{L^a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\]

- element load vector for constant $b_1 = b$

\[
r^a = \frac{\rho b L^a}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

- structural stiffness relation $Ku = r$

\[
\begin{bmatrix} \frac{E}{L} & -\frac{E}{L} \\ -\frac{E}{L} & \frac{E}{L} \end{bmatrix} \begin{bmatrix} u^1_1 \\ u^2_1 \end{bmatrix} = \frac{\rho b L}{2}
\]

- homogeneous DIRICHLET boundary condition

(deleting first line and first column)

\[
\begin{bmatrix} \frac{E}{L} \\ \frac{E}{L} \end{bmatrix} \begin{bmatrix} u^2_1 \end{bmatrix} = \frac{\rho b L}{2}
\]

- linear system of equations for $u^2_1$

solution identical to first finite element application
tension bar loaded by $\rho b_1$

$\rho b_1, b_1 = b = \text{constant}$

$E, \rho, A$

$L$

- element stiffness matrices, element lengths $L^e = L/2$

$$k^a = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k^b = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- element load vector for constant $b_1 = b$

$$r^a = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$r^b = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

fe discretization

two elements
tension bar loaded by $\rho b_1$

- element stiffness matrices, element lengths $L^e = L/2$
  
  $$k^a = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad k^b = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- element load vector for constant $b_1 = b$
  
  $$r^a = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad r^b = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- structural stiffness relation $Ku = r$

<table>
<thead>
<tr>
<th></th>
<th>$\frac{2E}{L}$</th>
<th>$-\frac{2E}{L}$</th>
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<th>$-\frac{2E}{L}$</th>
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<tbody>
<tr>
<td>$u^1_1$</td>
<td>$\frac{\rho b L}{4}$</td>
<td>$\frac{\rho b L}{4}$</td>
<td>$\frac{\rho b L}{4}$</td>
<td>$\frac{\rho b L}{4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

fe discretization

two elements
- Element stiffness matrices, element lengths $L^e = L/2$

$$
k^a = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad k^b = \frac{2E}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
$$

- Element load vector for constant $b_1 = b$

$$
r^a = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad r^b = \frac{\rho b L}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

- Structural stiffness relation $Ku = r$

$$
\begin{array}{ccc|c|c}
\frac{2E}{L} & -\frac{2E}{L} & \frac{1}{L} & u_1^2 & \frac{\rho b L}{4} \\
-\frac{2E}{L} & \frac{2E}{L} + \frac{2E}{L} & -\frac{2E}{L} & u_1^2 & \frac{\rho b L}{4} + \frac{\rho b L}{4} \\
-\frac{2E}{L} & \frac{2E}{L} & \frac{2E}{L} & u_1^3 & \frac{\rho b L}{4} \\
\end{array}
$$

- Homogeneous Dirichlet boundary condition

$$
\begin{array}{ccc|c|c}
\frac{4E}{L} & -\frac{2E}{L} & u_2^2 & \frac{\rho b L}{2} \\
-\frac{2E}{L} & \frac{2E}{L} & u_1^3 & \frac{\rho b L}{4} \\
\end{array}
$$
finite element model of coaxial tensile bar

- Fixed support: $u_1^1 = 0$
- Dirichlet displacement: $u_5^5 = u_1^*$

- Element 1

- Structural nodes and displacements (id)
  - Vertical: structural node numbers
  - Horizontal: spatial direction (here just 1)
  - Array: equation numbers dof’s
    - Positive: unknown displacements
    - Negative: prescribed displacements

- Element and structural nodes (it)
  - Vertical: element node numbers
  - Horizontal: element numbers
  - Array: structural node number
homework: In the present homework a finite element program for the static analysis of one dimensional continua should be developed using your favorite programming language. Therefore, linear ($p = 1$), one dimensional continuum elements should be applied. The correct implementation of the finite element and finite element procedure on the structural level should be verified by means of above sketched model problems. These examples are described by a truss loaded by a constant (load case i) and a linear (load case ii) line load. They should be analyzed using using $NE = 1, 2, 4, 8, 16, 32$ finite elements for the discretization of the truss. For these reasons the following working stages are proposed:

- develop a finite element routine for calculation of the element stiffness ’tensors’ $k^{eij}$ and the
In the present homework a finite element program for the static analysis of one dimensional continua should be developed using your favorite programming language. Therefore, linear ($p = 1$), one-dimensional continuum elements should be applied. The correct implementation of the finite element and finite element procedure on the structural level should be verified by means of above sketched model problems. These examples are described by a truss loaded by a constant (load case i) and a linear (load case ii) line load. They should be analyzed using $NE = 1, 2, 4, 8, 16, 32$ finite elements for the discretization of the truss. For these reasons the following working stages are proposed:

- develop a finite element routine for calculation of the element stiffness 'tensors' $k_{ij}$ and the consistent load 'tensors' $r_{ei}$ for $i, j \in [1, 2]$ for load case i
- develop a finite element procedure for the discretization with only one finite element $NE = 1$, implement an assembly procedure and the solution procedure for the linear system of equations with $NEQ = 1$. Check the correctness of the solution by means of load case i
- extend your finite element program for analyses with $NE = 2, 4, 8, 16, 32$ finite elements, yielding $NEQ = 1, 2, 4, 8, 16, 32$ unknowns, and check your solutions for load case i
- extend your finite element program for analyses of load case ii with $NE = 1, 2, 4, 8, 16, 32$ finite elements
- extend your finite element program by a post-processing procedure, calculating the approximations of the displacement $u_1$, stress $\sigma_{11}$ and residuum $\sigma_{11,1} + \rho b_1$
- calculate the local (at position $X_1$) and global (of the hole system) displacement errors with respect to the analytical solution
- plot diagrams of the displacements, stresses, the residuum and the local displacement error
- plot a double logarithmic diagram of the global error as function of the element length and, additionally, as function of the numbers of degrees of freedom of the system $NEQ$

The homework should be uploaded until December 19, 2013 on the moodle course. Your submission should include:

- a brief report documenting your results in form of diagrams
- and your program code